Problems in Mathematics
Snarks

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Abstract

In this essay we will demonstrate how the snark arose from the work of mathematicians studying the four-colour problem, note their rarity and briefly explore their construction.

Contents

Chapter 1

From map to snark

1.1 Introduction

We begin our story with the Four Colour Problem that, in its hundred-year intractability, played a significant part in driving the development of graph theory as a mathematical field. The problem was considered remarkable because it could "be explained in five minutes by any mathmatician to the so-called man in the street and at the end of the explanation, both would understand it, but neither would be able to solve it" [3].

This is probably still the case for many mathematicians today, since the accepted proof by Appel & Haken in 1977 was achieved by reducing the problem to hundreds of classes of graph "by hand" and using computers to analyse hundreds of thousands of individual cases [8, 9]. Much of the literature relating to graph theory (ie. all that written before 1977) refers to the "Four Colour Problem". However, due to Appel & Haken, it has been established that we may properly colour any map using four colours and as such may happily refer to the "Four Colour Theorem".

In this chapter we will explain, with simplified proof, why the statement that there are no planar snarks is equivalent to the statement of the Four Colour Theorem. Saaty [5] offers a good overview (including a timeline from 1852–1972) of many other equivalent statements of the Four Colour Theorem.

Note. In this chapter some effort is made to make clear to the reader whether we are discussing edge or vertex colourings. As such, both shall occasionally be referred to in their more verbose form, where the former is (proper) k-edge colouring and the latter is (proper) k-vertex colouring.

1.2 From map to graph

The problem that is the subject of the Four Colour Theorem (which we subsequently may abbreviate to 4CT) is defined in terms of maps, that is drawings that represent the division of the surface of a sphere into regions. This gives us an initial definition.

Definition 1.2.1. Given a **planar map**, say M, the **dual graph** of M, written $D(M) = G$ is the graph that has a vertex for every region of M and an edge between two vertices if and only if the corresponding regions share a border.

Whitney defines the dual graph or dual representation as the graph obtained from a planar map by "marking in each region of the map a point, which will be a vertex of the graph" and "across each boundary line of the map drawing a line connecting the vertices in the two regions the boundary separates, which will form an edge of the graph" [2]. According to Saaty [5], it follows from Whitney's definition of the dual graph that if the map M is planar, its dual graph $D(M) = G$ is also planar. From here we may begin to discuss the problem in terms of graph theoretic objects we know about from eg. the notes.

1.3 Refining the search

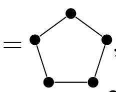
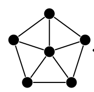
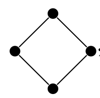
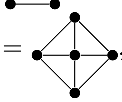
Before using Harary [4] to show directly the equivalence we desire, we will give a flavour of the thinking mathematicians were using in their attempts at proof for the 4CT. Both Tait and Whitney reduce the problem such that they need only consider **triangular graphs**. That is, a graph in which every face is a triangle. In their original papers, Tait describes a "diagram [with] three-sided compartments" and 50 years later Whitney describes a "graph composed of elementary triangles".

In his 1880 paper [1] Tait introduces a scheme where vertices are "intruded" into a triangular graph G such that it becomes a square graph G' (a graph where all faces have four sides), where G' is properly 2-vertex colourable. Since triangles have three sides, there is at least one further way of intruding vertices into the edges of G to produce a different square graph G'' which can also be properly 2-vertex coloured. Overlaying G' and G'' produces a vertex colouring for G that requires no more than 4 colours.

We will use Liu's 1968 presentation [3] of Tait's proof[1] here. Liu's formulation is slightly different and deals with a particular type of edge-colouring rather than Tait's vertex intrusion, but the ideas are similar despite the difference in language.

[1] Unfortunately I could only get the abstract for Tait's paper, so am not privy to the full proof or "rules laid down for carrying out [intrusion] operations".

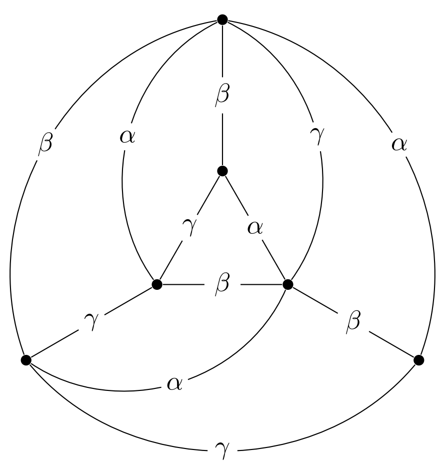
Definition 1.3.1. The **triangular transformation** of a planar graph G, $\mathcal{T}(G) = G'$ is obtained by dividing all the nontriangular faces of G into triangular faces. We can do this by introducing a vertex v in the centre of each nontriangular face and adding edges joining v to each of the vertices defining the face. G' is a **triangular graph**.

Example 1.3.1. The triangular transformation of $C_5 =$ , can be written $\mathcal{T}(C_5) =$ . For $C_4 =$ , we can write $\mathcal{T}(C_4) =$ , etc.

If a proper 4-vertex colouring of $\mathcal{T}(G)$ exists, then clearly this colouring is also a proper k-vertex colouring of G with $k \leq 4$. As such, at least for the remainder of this section, we need only consider triangular graphs (or the triangular transformations of nontriangular graphs).

Definition 1.3.2. We call a triangular graph G **triangle-edge colourable** if the edges of G can be coloured with three colours such that the edges of each face of G feature all three of these colours. We call such a colouring a **triangle-edge colouring**.

Example 1.3.2. An example of a triangular graph with a triangle-edge colouring on colours $\{\alpha, \beta, \gamma\}$.



Remark. A triangle-edge colouring is quite different to a 3-edge colouring, as we can see above, since here we consider the edges of each face independently.

Theorem 1.3.1. *(Tait in Liu [3] p. 253) A triangular graph $G = (V, E)$ has a proper 4-vertex colouring if and only if it is triangle-edge colourable.*

Proof. Suppose first that $\phi_{\mathcal{T}} : E \to \{\alpha, \beta, \gamma\}$ is a triangle-edge colouring of G, and consider the subgraph $H_1 = (V, E_1)$ where E_1 is the set of edges $e_1 \in E$ such that $\phi_{\mathcal{T}}(e_1) \in \{\alpha, \beta\}$. Since G is a triangular graph, our construction of H_1 means it will be a square graph and there will exist a proper 2-vertex colouring of H_1, say $\phi_{H_1} : V \to \{A, B\}$. We can construct a similar subgraph $H_2 = (V, E_2)$ where E_2 is the set of edges $e_2 \in E$ such that $\phi_{\mathcal{T}}(e_2) \in \{\alpha, \gamma\}$. Again, by construction we know there exists a proper 2-vertex colouring of H_2, let's say $\phi_{H_2} : V \to \{u, v\}$.

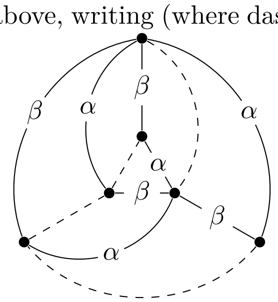
We can construct a function on the whole of G where each vertex is mapped to one of the four "mixed" colours in $\{A, B\} \times \{u, v\}$ by "superimposing" the vertex colourings ϕ_{H_1} and ϕ_{H_2}. Such a can be written $\phi : V \to \{A, B\} \times \{u, v\}$ where $\phi(x) = (\phi_{H_1}(x), \phi_{H_2}(x))$ for all $x \in V$.

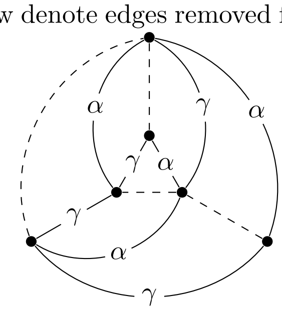
To see that the function ϕ is a proper 4-vertex colouring, suppose for $x, y \in V$ that x and y are the endvertices of some edge $e \in H_2$ such that $\phi_{H_1}(x) = \phi_{H_1}(y)$. By construction, we know that $\phi_{H_2}(x) \neq \phi_{H_2}(y)$ and it follows $\phi(x) \neq \phi(y)$. This is true for any pair of adjacent vertices in G and hence ϕ is a proper 4-vertex colouring of G. As such, G is properly 4-vertex colourable as required.

For the converse, see Liu [3] (p. 253).

□

Example 1.3.3. Using the triangular graph G and its triangle-edge colouring defined in Example 1.3.2, we construct example H_1 and H_2 subgraphs as described above, writing (where dashed lines below denote edges removed from G)

for H_1, 

and for H_2, 

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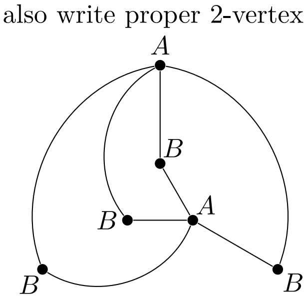
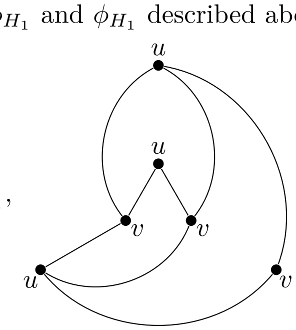
We can also write proper 2-vertex colourings ϕ_{H_1} and ϕ_{H_1} described above as,

for ϕ_{H_1}, 

and for ϕ_{H_1}, 

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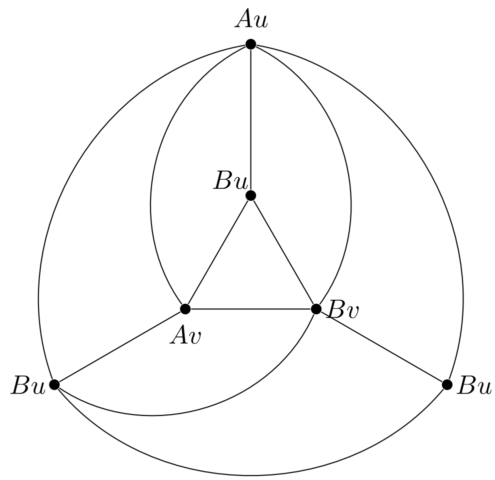
To complete the example we write the "superimposed" proper 4-vertex colouring, which we called ϕ, on the graph from Example 1.3.2 as



,

where we write eg. Au instead of (A, u) for legibility.

1.4 Arriving at the snark

In the previous section, we saw how mathematicians were able to find different ways of considering the map colouring problem by formulating equivalent problems in terms of graph theory. In this spirit, we will now demonstrate the promised result by using a theorem of Harary as a (somewhat lengthy) lemma and arrive at the subject of this essay, the snark.

Lemma 1.4.1. *(Harary [4] p. 132) The 4CT is equivalent to the statement that every bridgeless planar cubic graph is properly 4-vertex colourable (henceforth 4-colourable).*

Proof. We have seen (by Definition 1.2.1) that the 4CT is equivalent to the statement that every planar graph is 4-colourable. Considering the map whose dual graph has a bridge, we note that contracting (identifying the endvertices of) the bridge in the dual graph does not affect the number of regions or their adjancency in the corresponding map. As such we know that the 4CT is equivalent to the statement that every planar bridgeless graph is 4-colourable. Certainly if every bridgeless planar graph is 4-colourable then every bridgeless *cubic* planar graph is 4-colourable.

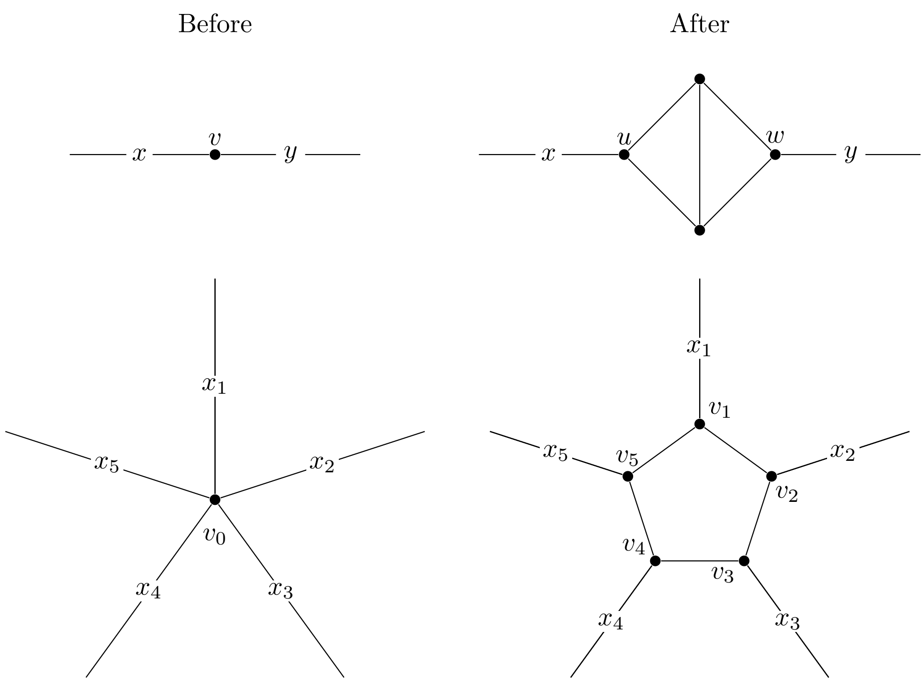
For the converse, let G be a bridgeless planar graph and assume that all bridgeless cubic planar graphs are 4-colourable. We will show a method of constructing a cubic graph from G.

Since G is bridgeless we know for any vertex v of G that $\deg(v) \geq 2$. Similarly, if G contains a vertex v of degree 2 incident with edges x and y, we subdivide x and y, denoting the subdivision vertices as u and w respectively. We now remove v, identify u with one of the vertices of degree 2 in a copy of the graph $K_4 - e$ and identify w with the other vertex of degree 2 in $K_4 - e$. Observe that each of the vertices so added has degree 3. If G contains a vertex v_0 of degree $n \geq 4$ incident with edges $x_1, x_2, ..., x_n$ arranged cyclically about v_0, we subdivide each x_i introducing a new vertex v_i. We then remove v_0 and add new edges $v_1v_2, v_2, v_3, ..., v_{n-1}v_n, v_nv_1$. Again, each of the vertices so added has degree 3.

Denote the bridgeless cubic planar graph resulting from this process as G'. Clearly identifying in G' all vertices in $G' - G$ (ie. those added under the above scheme), we arrive back at G. By hypothesis there is a 4-colouring of G' and indentifying vertices as described above induces a k-colouring of G with $k \leq 4$, which completes the proof.[2]

□

Example 1.4.1. The process described above is easily understood using pictures. Following the notation of the previous theorem, for vertices of degree 2 and $n \geq 4$ we can see how each is replaced by vertices of degree 3.



[2]I found Harary's wording of the main body of this proof very clear, so reproduce it here.

Having demonstrated that the 4CT is equivalent to the statement that every bridgeless cubic planar graph is properly 4-vertex colourable, we are closing in on the snark. We use another theorem of Harary to bring it it within our grasp.

Theorem 1.4.1. *(Harary [4] p. 134) The 4CT is equivalent to the statement that every bridgeless cubic planar graph is 3-edge colourable.*

Proof. We have seen by Lemma 1.4.1 that the 4CT is equivalent to the statement that every bridgeless cubic planar graph is properly 4-vertex colourable (or simply 4-colourable).

First assume $G = (V, E)$ is a bridgeless cubic planar graph with a 4-vertex colouring $\phi_V : V \to K$ where K is the set of elements of the Klein four-group where addition is defined $k_i + k_i = k_0$ and $k_1 + k_2 = k_3$ with k_0 the identity element. Let $\phi_E : E \to K \setminus \{k_0\}$ be an edge-colouring of G defined by the group sum $\phi_E(e) = \phi_V(x) + \phi_V(y)$ where x and y are the endvertices of e an edge in G. It is immediate by the properness of ϕ_V that ϕ_E is a 3-edge colouring[3] and hence the chromatic index of G is three, as required[4].

For the converse, first let G be a bridgeless cubic planar graph with chromatic index 3 and colour it with eg. ϕ_E as above. To colour the "regions" of G (considering it as a planar map as in Definition 1.2.1) select a region R_0 and colour it k_0. To colour an arbitrary region R, draw a curve C from the interior of R_0 to the interior of R that doesn't pass through any vertex of G and assign the colour to be the sum under group addition of $\phi_E(e)$ for all edges e incident with C. Call this colouring ϕ_R.

To see that the colours assigned to regions under this scheme will be the same regardless of our choice of C and only depends on our choice of R_0 (ie. that this colouring is well-defined), consider a closed simple curve S that doesn't pass through a vertex of G. Observe that if $c(v)$ denotes the the sum of the colours of edges incident with a vertex v, then $c(v) = k_1 + k_2 + k_3 = k_0$. Let S_V be the set of vertices interior to the closed curved S, now we have $\sum_{v \in S_V} c(v) = k_0$. Let $c_1, .., c_n$ be the colours of the edges incident with S and let $d_1, ..., d_m$ be the colours of the edges interior to S. Because every edge for a d_i is incident with exactly 2 vertices in S_V and every edge for a c_i is incident with exactly 1 vertex in S_V, under Klein four-group addition we know that

$$\sum_{v \in S_V} c(v) = c_1 + \cdots + c_n + 2(d_1 + \cdots + d_m) = c_1 + \cdots + c_n = k_0.$$

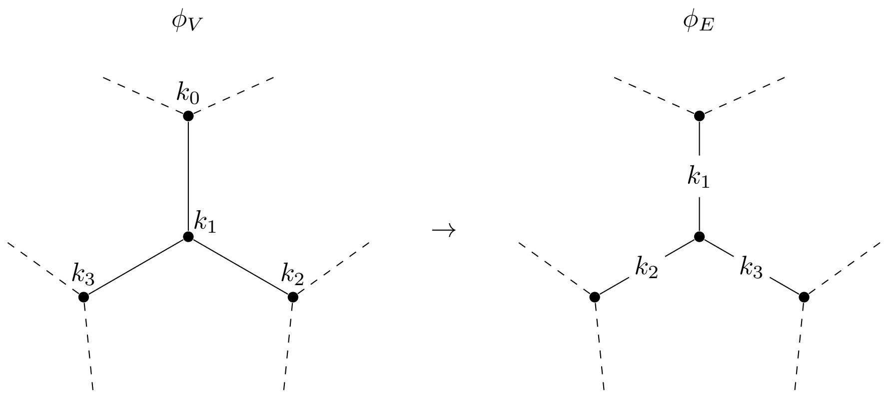
To complete the proof, note that since under eg. ϕ_E no edge is coloured k_0, no two regions sharing a border will be assigned the same colour under ϕ_R.

□

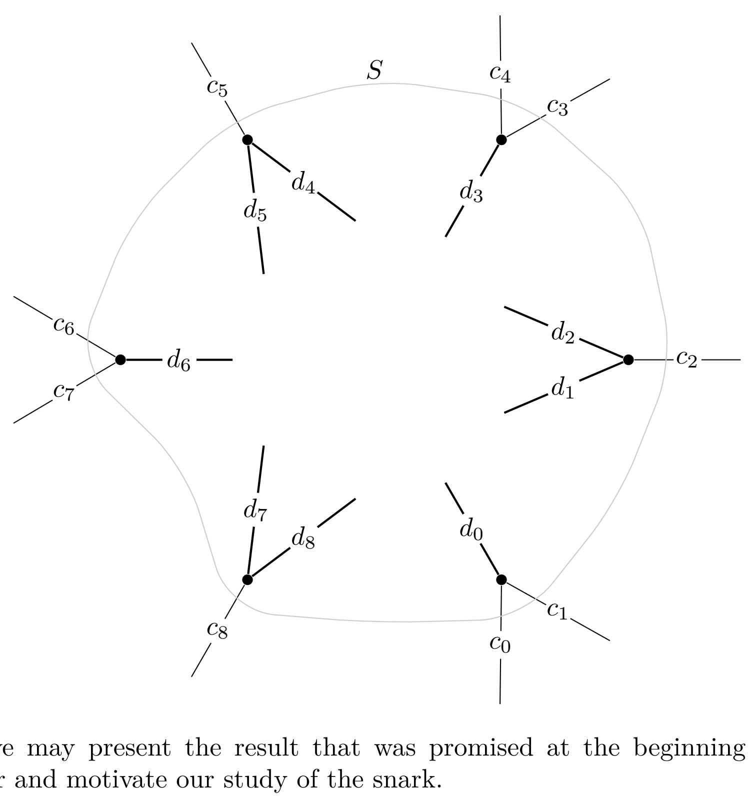
[3]By hypothesis ϕ_V is a proper vertex colouring, so any adjacent vertices u and v will have the property $\phi_V(u) \neq \phi_V(v)$ and hence $\phi_V(u) + \phi_V(v) \neq k_0$.

[4]Since G is cubic we know $\chi'(G) \geq 3$ and having shown there is a 3-edge colouring of G we know $\chi'(G) \leq 3$ hence $\chi'(G) = 3$.

Example 1.4.2. The picture below shows an example of how vertex colouring ϕ_V maps to edge colouring ϕ_E in the proof of Theorem 1.4.1.



Example 1.4.3. To clarify the Klein group sum used in the proof of Theorem 1.4.1, we can illustrate the incidence and interiority of edges (interior, twice counted edges coloured d_i are drawn thick) and vertices in relation to the closed curve S (drawn light grey) as follows.



Now we may present the result that was promised at the beginning of this chapter and motivate our study of the snark.

Corollary 1.4.1. *The 4CT is equivalent to the statement that there are no planar snarks.*

Proof. This follows immediately from Theorem 1.4.1. A planar snark would be a bridgeless cubic planar graph that was not 3-edge colourable, a contradiction.

□

Chapter 2

The rarity of the snark

Since the naming of the snark by Gardner[1] in [7] its defintion has undergone some refinement (eg. [6], [11], [12], [13]) in order to preclude trivial examples from snarkdom. To talk more precisely about the snark, we use recently written definitions from the 2012 paper of Brinkmann, Goedgebeur, Hägglund & Markström [13]. In these definitions, a graph G is **uncolourable** if $\chi'(G) = 4$ (as in [6]).

2.1 Definitions

Definition 2.1.1. The **girth** of a graph is the number of vertices in a shortest cycle in G and is denoted $g(G)$.

Definition 2.1.2. A **digon** is a 2-cycle.

Definition 2.1.3. A graph G is **cyclically k-edge connected** if the deletion of fewer than k edges from G does not create two components both of which contain at least one cycle. The largest integer k such that G is cyclically k-edge connected is called the **cyclic edge-connectivity** of G and is denoted $\lambda_c(G)$.

Definition 2.1.4. A **weak snark** is an uncolourable cyclically 4-edge connected cubic graph with girth at least 4. A **snark** is an uncolourable cyclically 4-edge connected cubic graph with girth at least 5.

We further define a **trivial snark** to be the snark that we arrived upon at the end of the previous chapter in Corollary 1.4.1 (ie. an uncolourable cubic graph without restrictions on cyclic connectivity or girth). Note that all snarks are weak snarks and all weak snarks are trivial snarks.

In the preceding definitions, restrictions are placed on the girth and cyclic edge connectivity of a graph for it to be considered a snark and not (merely) a weak or trivial snark. In the following two sections we will examine and

[1] Gardner incidentally edited an annotated version of Carroll's poem.

justify these conditions which originate in Isaacs [6] where "flower snarks" are also introduced. We consider the construction of flower snarks in the following chapter, Chapter 3.

2.2 Girth

As with connectivity (covered in Section 2.3) the motivation for putting the lower bound on the girth of a snark G to $g(G) > 4$ can be seen in some sense as an effort to preserve their rarity. In this section we shall see that snarks which are trivial by virtue of their girth can always be **reduced** to a nontrivial snark by performing simple operations.

Isaacs showed in 1975 that a trivial snark $|G| = n$ with digons, triangles or squares can easily be altered to produce a snark $|H| < n$. Such a G is "trivially similar" to the simpler snark H.

Lemma 2.2.1. *Let G be a trivial snark with $g(G) = k$. We can always construct an H with a subset of G's vertices where $\chi'(H) = 4$.*

Proof. We will demonstrate individually for each of $k \in \{4, 3, 2\}$.

$k = 2$: Identify the vertices of the digon to a vertex v and then identify v with either one of its neighbours.

$k = 3$: Apply the reverse process of that described in Theorem 1.4.1 (ie. the identification of the vertices of the triangle).

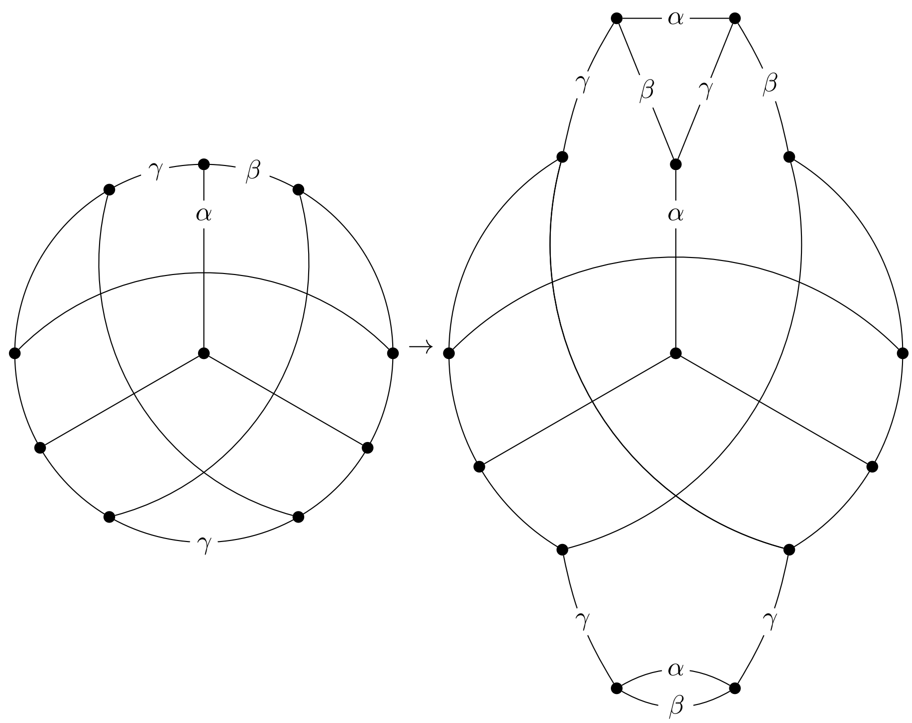
$k = 4$: Remove the vertices of the square, leaving four vertices of degree two. Add two edges between pairs of these vertices in such a way that all have degree three. This case is also mentioned in Isaacs [6] (2.4.1).

It is sufficient for this proof to simply remove digons, triangles and squares, since, being themselves colourable, none of these parts of the graph is that which induces the graph as a whole to be uncolourable.

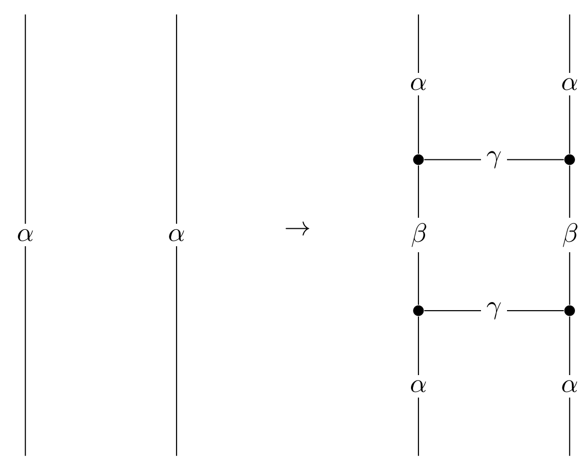
□

Similarly, it is easy to add digons, triangles and squares to a snark to produce arbitrarily many trivial snarks. Clearly any edge of a snark can have a digon introduced to it without causing the graph to become colourable. Similarly, since the snark is cubic, the process described in Lemma 1.4.1 can by applied to any vertex to produce a triangle. Example 2.2.2 shows how one might add a square to a snark without affecting its uncolourability.

Example 2.2.1. We can form a trivial snark by adding a digon and a triangle to a 4-edge coloured Petersen graph, a nontrivial snark. Clearly, reversing this process amounts to the reduction of a trivial snark as described above.



Example 2.2.2. To add a square to a snark, select two edges that bear the same colour, in each introduce two vertices, connect them pairwise. It is easy two see this configuration will be edge colourable using three colours.



2.3 Connectivity

Another idea introduced by Isaacs to limit what should be considered a snark is that of decomposition, where a k-cyclic edge connected snark is shown to decompose into at least one simpler snark when $k \leq 3$. We present his proof in modified language that the decomposition of a trivial snark yields at least one simpler snark.

Definition 2.3.1. Let G be a trivial snark with cyclic connectivity $k \leq 3$. We will **decompose** G into two cubic graphs G_0 and G_1. If G has a 2-edge cut, take it and add an edge to each component connecting the two vertices of degree two, consider each component a graph, G_0 and G_1. If G has a 3-edge cut, take it and for each component add an isolated vertex v_0, v_1. Join vertices in each component having degree two to v_i and consider each component to be a graph G_0, G_1. The graphs G_0 and G_1 are the **decomposition** of G.

Isaacs shows in [6] (2.3.4) that every colourable graph G (with colouring ϕ) has a hamiltonian cycle, and if G has a 2-edge cut those edges will have the same colour under ϕ. It is also shown that if G has a 3-edge cut, its edges will take distinct colours under ϕ. This result is not proved here, but is used in the proof below.

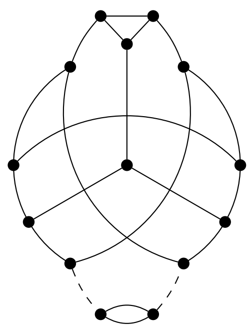
Lemma 2.3.1. *Any trivial snark G with cyclic connectivity $k \leq 3$ has a decomposition G_0, G_1 at least one of which is a trivial snark.*

Proof. First suppose G has a colouring, say ϕ. If G has a 2-edge cut its edges will have the same colour, say α. Let e and e' be the edges added in the decomposition of G and set $\phi(e) = \phi(e') = \alpha$ and both G_0 and G_1 are colourable under ϕ. If G has a 3-edge cut, say $\{xx', yy', zz'\}$, and its decomposition introduces edges $\{xa, ya, zy\}$, since we know $\phi(xx') \neq \phi(yy') \neq \phi(zz')$ we can set $\phi(xa) = \phi(xx')$, $\phi(ya) = \phi(yy')$ and $\phi(za) = \phi(zz')$ to make ϕ colour the decomposition of G.

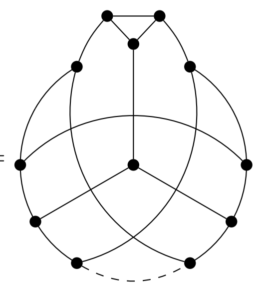
Conversely, suppose that G_0 and G_1 arising from the decomposition of G are coloured under ϕ_0 and ϕ_1. Relabelling the colours in ϕ_i (by [6] 2.3.4) we can use it as a colouring for G. Hence, if G is not colourable then its decomposition is not colourable.

□

Example 2.3.1. We show the decompositions of the "trivialised" Petersen graph used in Example 2.2.1 to return to a nontrivial snark, first by taking a 2-edge cut we write (where dashed lines denote the edge cut)



$\rightarrow$ $G_0 =$



, $G_1 =$

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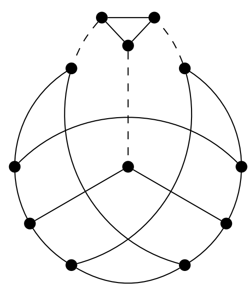
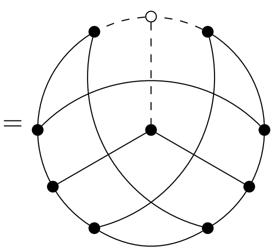
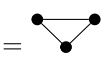
Subsequently taking the 3-edge cut of $H = G_0$ we write a second decomposition



$\rightarrow$ $H_0 =$



, $H_1 =$



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Clearly H_0 is the Petersen graph. Note that the decompositions above are trivial in themselves, since reduction would produce the same result.

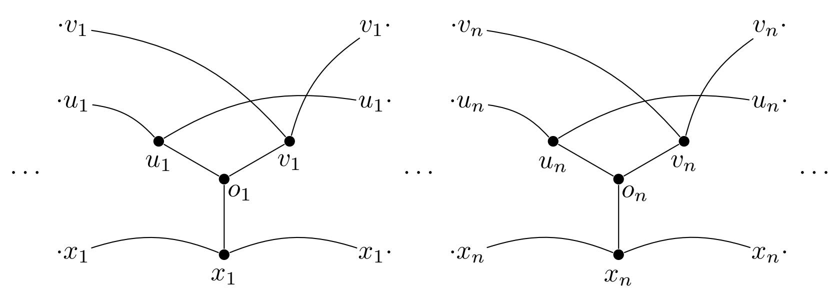
We have seen how removing particular configurations of edges and vertices from snarks allows us to better consider what makes the snark uncolourable. In the next chapter, we will consider precisely such an example of a generally uncolourable configuration.

Chapter 3

Generating snarks

We conclude by considering the construction of the infinite family of snarks discovered by Isaacs in 1975, the flower snarks. This construction demonstrates how isolating a particular configuration of edges and vertices which can be "composed" with copies of itself in such a way that the resulting graph will always be uncolourable.

Isaacs construction is simple and uses a graph fragment (which he calls a "pendant graph" [6]), two copies of which are depicted below[1].



An odd number of such fragments are arranged cyclic fashion where all but one pairs are joined in the obvious way (ie. edges are u_iu_{i+1}, v_iv_{i+1}, etc), the pair completing the cycle being connected by edges x_nx_1, u_nv_1 and v_nu_1. Isaacs' proof that such a graph is uncolourable is also simple.

Proof. We consider, when $\cdot u_i$, $\cdot v_i$ and $\cdot x_i$ are coloured a particular way, the possibilities for the colours of $u_i\cdot$, $v_i\cdot$ and $x_i\cdot$. We will call these **incolours** and **outcolours** respectively and denote each case with an ordered triple from $\{a, b, c\}$. Having either incolours or outcolours (a, a, a) is impossible since one of the edges v_io_i, u_io_i, x_io_i must be coloured a. A pattern of incolours with two colours (one featuring twice) produces outcolours where the third colour features twice. For example (a, a, b) can give either (c, b, c) or (b, c, c). This

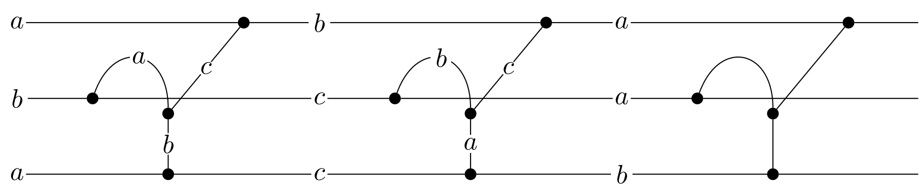
[1] We use the notation $u\cdot$ and $\cdot u$ to mean an edge incident with vertex u, connected to an as-yet-undecided vertex.

alternates as we proceed with construction, so an odd number of composed fragments leads to edges with the twice-featuring colour meeting at the same vertex, thus our graph cannot be coloured in this way. Lastly, incolours (a, b, c) give outcolours that are a rotation of the elements of the triple that preserves their order, but by the $u_n v_1$ and $v_n u_1$ "crossover" edges that complete our cycle of graph fragments, these incolours also give rise to uncolourability.

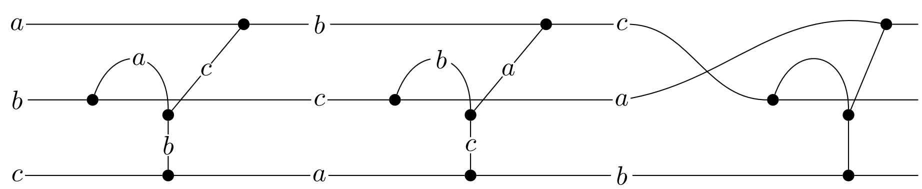
Therefore, a graph constructed as described is not colourable.

□

Example 3.0.1. (a, a, b) incolours



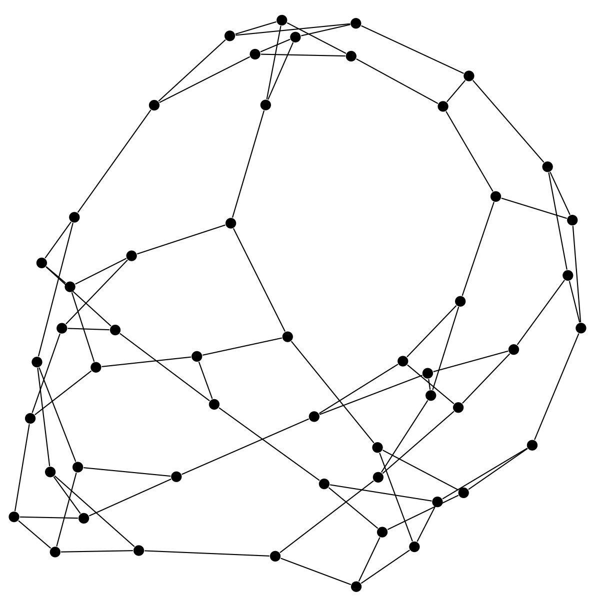
Example 3.0.2. (a, b, c) incolours



Chapter 4

Conclusion

We have seen how the snark arose naturally from the study of map colouring problems and was named just as the 4CT was proved. We have also seen how the snark has established itself as a mathematical object worthy of study in its own right, with refinements to its definition following discoveries about its properties. We have considered Isaacs' construction of an infinite family of snarks. Future study relating to reducibility, etc. of the snark may yield further infinite families and a deeper understanding of how graph structure relates to colourability.



A snark on 44 vertices [14]

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